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PARTIAL DIFFERENTIAL EQUATIONS: PART-1

Lecture #13

Partial Differential Equation (PDE)

Introduction

A partial differential equation is a relationship between a dependent variable (u) and two or more independent variables (x, y, t,...) and partial derivatives of u with respect to these independent variables. The solution is therefore of the form $u = f(x, y, t, \ldots)$.

PDE are models for various physical systems such as in dynamics, elasticity, heat transfer, electromagnetic theory and quantum mechanics.

Partial Differential Equation (PDE)

Linear Partial Differential Equation

The general form of a linear second order PDE is given by:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G,$$

When $G(x, y) = 0$, to be **homogeneous;** otherwise, it is **nonhomogeneous**

Example:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = xy$$

homogeneous

nonhomogeneous.

Partial Differential Equation (PDE)

Important Linear Second Order PDEs

(1)	$\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$	*One-dimensional wave equation*
(2)	$\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$	*One-dimensional heat equation*
(3)	$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$	***Two-dimensional Laplace equation***
(4)	$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x, y)$	***Two-dimensional Poisson equation***
(5)	$\dfrac{\partial^2 u}{\partial t^2} = c^2 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$	***Two-dimensional wave equation***
(6)	$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$	***Three-dimensional Laplace equation***

Here c is a positive constant, t is time, x, y, z are Cartesian coordinates, and *dimension* is the number of these coordinates in the equation.

Partial Differential Equation (PDE)

Initial conditions and Boundary conditions

As with any differential equation, the arbitrary constants or arbitrary functions in any particular condition are determined from the additional information given. These are called the initial conditions or, more generally, the boundary conditions since they do not always refer to zero values of the independent variables.

Solution of a PDE

It is not our intention to examine the general procedure to find the *general solutions* of linear PDE. It is often difficult to obtain and not all useful in applications. Thus the focus will be on finding *particular solutions* of some important linear PDEs that appear in many applications.

Partial Differential Equation (PDE)

Solution by direct integration

The simplest form of PDE is such that a solution can be determined by direct partial integration those similar to ODE.

Example 1

Solve the equation $\frac{\partial^2 u}{\partial x^2} = 12x^2(t+1)$ given that at $x = 0$, $u = \cos\ 2t$ and $\frac{\partial u}{\partial x} = \sin t$. Notice that the initial conditions are functions of t and not just constants.

$$u(0,t) = \cos 2t \quad \text{and} \quad u'(0,t) = \left.\frac{du}{dx}\right|_{x=0} = \sin t$$

Partial Differential Equation (PDE)

Solution →

Integrating with respect to *x*: $\frac{du}{dx} = 4x^3(t+1) + \phi(t)$

Arbitrary function $\phi(t)$ takes place of arbitrary constant in ODE.

Integrating w.r.t. *x* again: $u = x^4(t+1) + x\phi(t) + \theta(t)$

To find the two arbitrary functions $\phi(t)$ and $\theta(t)$, we apply the initial conditions into two equations above,

$$u(0,t) = \cos 2t \quad \text{and} \quad \left.\frac{du}{dx}\right|_{x=0} = \sin t$$

We get:

$\sin(t) = \phi(t)$ and $\cos(2t) = \theta(t)$

Thus,

$$u(x,t)=x^4(t+1)+x\sin(t)+\cos(2t)$$

Partial Differential Equation (PDE)

Example 2

Solve the equation: $\frac{d^2u}{dxdy} = \sin(x+y)$, given that:

$\left.\frac{du}{dx}\right|_{y=0} = 1$ and $u(0,y) = (y-1)^2$

Solution →

Integrating with respect to *y*: $\frac{du}{dx} = -\cos(x+y) + \phi(x)$

$$\left.\frac{du}{dx}\right|_{y=0} = 1 \quad \rightarrow 1 = -\cos(x) + \phi(x) \rightarrow \phi(x) = 1 + \cos(x)$$

$$\rightarrow \frac{du}{dx} = -\cos(x+y) + 1 + \cos(x)$$

Integrating with respect to *x*:

$$u = -\sin(x+y) + x + \sin(x) + \theta(y)$$

$$u(0,y) = (y-1)^2 \rightarrow (y-1)^2 = -\sin(y) + \theta(y) \rightarrow$$

$$\theta(y) = \sin(y) + (y-1)^2$$

$$u = -\sin(x+y) + x + \sin(x) + \sin(y) + (y-1)^2$$

Partial Differential Equation (PDE)

Example 3

Solve the equation $\dfrac{\partial^2 u}{\partial x \partial y} = \sin x \cos y$, subject to the boundary conditions that at $y = \dfrac{\pi}{2}$, $\dfrac{\partial u}{\partial x} = 2x$ and at $x = \pi$, $u = 2\sin y$.

Work through it: it is easy enough. $u = \ldots\ldots\ldots\ldots$

Solution →

$$\boxed{u = x^2 + \cos x(1 - \sin y) + \sin y + 1 - \pi^2}$$

Because

$$\frac{\partial^2 u}{\partial x \partial y} = \sin x \cos y \qquad \therefore \quad \frac{\partial u}{\partial x} = \sin x \sin y + \phi(x)$$

$$\text{But } \frac{\partial u}{\partial x} = 2x \text{ at } y = \frac{\pi}{2} \qquad \therefore \quad \phi(x) = 2x - \sin x$$

$$\therefore \quad \frac{\partial u}{\partial x} = 2x - \sin x(1 - \sin y) \qquad \therefore \quad u = x^2 + \cos x(1 - \sin y) + \theta(y)$$

But $u = 2\sin y$ at $x = \pi$ $\quad \therefore \quad \theta(y) = 1 - \pi^2 + \sin y$

$$u = x^2 + \cos x(1 - \sin y) + \sin y + 1 - \pi^2$$

Partial Differential Equation (PDE)

Solution by separation of variables

In the method of separation of variables, we seek to find a particular solution of the form of a *product* of a function x and a function of y.

$$u(x, y) = X(x)Y(y)$$

$$\frac{\partial u}{\partial x} = X'Y, \quad \frac{\partial u}{\partial y} = XY', \quad \frac{\partial^2 u}{\partial x^2} = X''Y, \quad \frac{\partial^2 u}{\partial y^2} = XY'',$$

Example 4

Find product solutions of $\dfrac{\partial^2 u}{\partial x^2} = 4\dfrac{\partial u}{\partial y}$.

SOLUTION

$u(x, y) = X(x)Y(y)$

$$\frac{\partial u}{\partial x} = X'Y, \quad \frac{\partial u}{\partial y} = XY', \quad \frac{\partial^2 u}{\partial x^2} = X''Y, \quad \frac{\partial^2 u}{\partial y^2} = XY'',$$

$$\Rightarrow \quad X''Y = 4XY'$$

After dividing both sides by $4XY$,

$$\Rightarrow \quad \frac{X''}{4X} = \frac{Y'}{Y}$$

In practice it is *convenient* to write this real **separation constant** as $-\lambda$

$$\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda$$

Since if they were variable (not a constant), changing x or y would affect only one side, leaving the other unaltered.

we obtain the two linear ordinary differential equations

$$X'' + 4\lambda X = 0 \quad \text{and} \quad Y' + \lambda Y = 0.$$

Then there are three cases for λ : zero, negative or positive.

CASE I If $\lambda = 0$,

$X'' = 0$ and $Y' = 0$. ← Double roots of Aux. Eq.

$$X = c_1 + c_2 x \text{ and } Y = c_3$$

$$\Rightarrow u = XY = (c_1 + c_2 x)c_3 = A_1 + B_1 x$$

CASE II If $\lambda = -\alpha^2$,

$X'' - 4\alpha^2 X = 0$ and $Y' - \alpha^2 Y = 0$. ← Distinct roots of A.E.

$$X = c_4 e^{2\alpha x} + c_5 e^{-2\alpha x} \quad \text{and} \quad Y = c_6 e^{\alpha^2 y}$$

$$\Rightarrow u = XY = (c_4 e^{2\alpha x} + c_5 e^{-2\alpha x})c_6 e^{\alpha^2 y} \quad \text{or}$$

$$u = XY = (A_2 e^{\alpha^2 y} e^{2\alpha x} + B_2 e^{\alpha^2 y} e^{-2\alpha x})c_6$$

where $A_2 = c_4 c_6$ and $B_2 = c_5 c_6$.

CASE III If $\lambda = \alpha^2$,

$X'' + 4\alpha^2 X = 0$ and $Y' + \alpha^2 Y = 0$ ← Complex roots of A.E.

$$X = c_7 \cos 2\alpha x + c_8 \sin 2\alpha x \quad \text{and} \quad Y = c_9 e^{-\alpha^2 y}$$

$$\Rightarrow u = A_3 e^{-\alpha^2 y} \cos 2\alpha x + B_3 e^{-\alpha^2 y} \sin 2\alpha x,$$

where $A_3 = c_7 c_9$ and $B_2 = c_8 c_9$.

References:

1. Advanced Engineering Mathematics, E.Kreyszig, 8th ed.
2. Advanced Engineering Mathematics, K.A.Stroud and D.J.Booth, 4th ed.

NEXT TOPICS

- One-dimensional wave equation (vibration of string) .
- One-dimensional heat equation.